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Clifford Algebra and the Propagation of Kähler Spinors

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The Kähler equation for an inhomogeneous differential form is analyzed in some detail and expressed in a set of coordinates called Riemann normal coordinates. A class of solutions to the Kähler spinors is constructed. It is shown how we can perturbatively decouple the Kähler equation and write its solution as a sum of spinors by considering the isomorphism between Clifford and the total matrix algebras.

1. INTRODUCTION

It is well known that the general theory of relativity predicts that gravitation manifests itself as a curvature of space-time. This curvature is characterized by the Riemann tensor R^i_{jkl}. Parker (1980; Parker and Pimentel, 1982) showed how the curvature of space-time at the position of an atom affects its spectrum. The frequency shifts caused by the curvature are different for various spectral lines, and in the Schwarzschild geometry the level spacing of the gravitational effect is different from that of the well-known first-order (*degenerate*) Stark and Zeeman effects. Manasse and Misner (1963) have studied the problem of two bodies in general relativity, in which one of the bodies is of small mass and, under the influence of gravitational attraction, moves toward a much larger mass whose field produces deformations in the geometry of the small one. The Manasse and Misner analysis of this problem is obtained from the metric of the background field, and is expressed in a set of comoving coordinates, called Fermi normal coordinates.

Recently, there has been some interest in using Clifford algebra in physics, one of the most well-known applications of this algebra being in

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the Dirac equation. The quantum field theory of free massive Dirac spin-1/2 particles has been obtained (Cho *et al.*, 1990) by using the method of two successive Clifford algebra constructions. The applications of the Clifford algebra involving spinors and in particular Dirac spinors have also been studied (Crawford, 1991).

With the incorporation of gravitational interactions (Modanese, 1992; Manko and Sibgatullin, 1992; Fradkin and Shvartsman, 1992) with other fields it became necessary to sharpen the notion of a spinor considerably since the gravitational effects according to Einstein were attributed to a nonflat space-time manifold. Such a manifold is said to carry a spinor structure (Hitchin, 1974; Chevalley, 1954; Penrose and Rindler, 1987) if the bundle of orthonormal frames with Lorentz structure group can be globally lifted into a bundle of spinor frames carrying a structure group that covers the Lorentz group.

In this paper we are mainly interested in studying the Kähler field equation (Kähler, 1962) and examine the propagation of its spinors using Riemann normal coordinates (RNC). The advantages of using such coordinates come from the fact that the connection coefficients $\Gamma^{\mu}_{\nu\rho}$ vanish at a point, while the essential part in constructing RNC is that the higher derivatives of the connection coefficients do not vanish at that particular point, thereby simplifying the computations of Riemann tensor, Einstein equations, etc. In this work we have found that writing the Kähler equation in terms of RNC makes it possible for us to perturbatively decouple this equation and expand its solution as a power series in the geodesic distance. This in turn will enable us to study the individual propagation of the Kähler spinors in curved space-time.

The Kähler equation for an inhomogeneous differential form, that is, a general element of the exterior algebra generated by a basis for the cotangent space of the space-time manifold (M), offers in Minkowski space an alternative description of half-integral spin to that provided by the conventional formulation of the Dirac equation. Apart from a penetrating paper by Graf (1978), this equation remained dormant until around 1982, when it was simultaneously taken up by a number of authors. Many were interested in using it to describe fermions on a lattice (Becher and Joos, 1982; Rabin, 1982; Banks *et al.*, 1982). The properties of the Kähler equation as a quantum field theory have been examined (Benn and Tucker, 1983). Fundamental questions like Lorentz invariance, electromagnetic coupling, breaking of degeneracy, and quantization have been discussed (Basarab-Horwath and Tucker, 1986). Recently, we have examined the local and global dynamics of the Kähler field equation and an algebraic spinorial solution to this equation lying in a minimal left ideal characterized by a certain idempotent projector has been analyzed (Talebaoui, 1993, 1994). In Minkowski space-time this equation decouples into four minimal left ideals of the space-time Clifford algebra, and it is equivalent to four identical Dirac equations. On the other hand, the Kähler equation might be more appropriate when discussing spinor fields and their relation to gravitation.

The vector space isomorphism between the exterior and Clifford algebras has been observed and exploited (Salingaros and Dresden, 1979; Salingaros, 1982; Budinich and Dabrowski, 1985; Ablamowicz and Salingaros, 1985; Ablamowicz *et al.*, 1982) by many physicists since the work of Chevalley. When the Clifford algebra is associated with a metric on a (pseudo-) Riemannian manifold, one may construct a useful calculus for the study of physical field theories involving gravitation. In this paper we shall exploit the isomorphism (Brihaye *et al.*, 1992; Finkelstein and Rodriguez, 1986; Kawamoto and Watabiki, 1992; Maia *et al.*, 1990) between the real Clifford algebra $C_{3,1}(R)$ and the total matrix algebra M_4, i.e., the algebra of 4×4 real matrices, and expand the Kähler field in a matrix basis. Also, we will show how one can perturbatively decouple the Kähler equation and write its solution as a sum of algebraic spinors (*elements of minimal left ideal*).

2. RIEMANNIAN NORMAL COORDINATES

In curved space-time one can never find a coordinate system with the coefficients $\Gamma^{\mu}_{\nu\rho} = 0$ everywhere (μ, ν, ρ = 0, 1, 2, 3). But one can always choose a coordinate system x^{μ} so that at any chosen space-time event, $\Gamma^{\mu}_{\nu\rho} = 0$. A very special and useful realization of such coordinates is a Riemannian normal coordinates system (Eisenhart, 1949; Dolgov and Khriplovich, 1983; Ni and Zimmerman, 1978; Kobayashi and Nomizu, 1963, 1969).

By a RNC system at x of a Riemannian manifold M, we mean a RNC system $x^1, \ldots, x^n$ at x such that $\partial/\partial x^1, \ldots, \partial/\partial x^n$ form an orthonormal frame at x. However, $\partial/\partial x^1, \ldots, \partial/\partial x^n$ may not be orthonormal at other points.

Let us now introduce the coordinate system (ξ^0, ξ^1, ξ^2, ξ^3; t) by

$$x^{\mu} = \xi^{\mu} t \tag{1}$$

$$\xi^{\mu}\xi_{\mu} = -\lambda \tag{2}$$

λ could take the value ± 1 or 0, and t is the geodesic distance. The duals X_{μ} to the orthonormal 1-forms in RNC are given by (Talebaoui, n.d.)

$$X_0 = \lambda\xi^0 \frac{\partial}{\partial t} - \frac{\xi^0}{t} \lambda\xi^n \frac{\partial}{\partial \xi^n} \quad \text{and} \quad X_s = -\lambda\xi^s \frac{\partial}{\partial t} + \left(\frac{\delta^{ns} + \lambda\xi^n\xi^s}{t}\right) \frac{\partial}{\partial \xi^n} \tag{3}$$

where n, s = 1, 2, 3. Using (3), we obtain

$$
\begin{aligned}
X_0(\xi^0) &= -\frac{\lambda}{t}(\underline{\xi})^2 & \qquad d\xi^0(X_0) &= -\frac{\lambda}{t}(\underline{\xi})^2 \\
X_0(\xi^n) &= -\frac{\xi^0}{t}\lambda\xi^n & d\xi^0(X_n) &= \frac{\lambda}{t}\xi^0\xi^n \\
X_n(\xi^0) &= \frac{\lambda}{t}\xi^0\xi^n & d\xi^n(X_0) &= -\frac{\lambda}{t}\xi^0\xi^n \\
X_n(\xi^s) &= \frac{1}{t}(\delta^{ns} + \lambda\xi^n\xi^s) & d\xi^n(X_s) &= \frac{1}{t}(\delta^{ns} + \lambda\xi^n\xi^s)
\end{aligned}
\tag{4}
$$

where $\underline{\xi}^2 \equiv (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2$. Equations (3) and (4) are essential in our forthcoming calculations.

3. THE KÄHLER EQUATION

The idea that differential forms can be used to describe fermions follows from a suggestion due to Kähler. He studied the general inhomogeneous differential form on space-time and used a correspondence between Clifford and exterior algebras associated with space-time to describe particles with half-integer spin by means of sections of the exterior bundle over space-time (*inhomogeneous differential forms*). Kähler's equation requires the use of inhomogeneous differential forms, whereas familiar theories of bosons, for example, the Maxwell or Klein–Gordon theories, require only homogeneous p-forms. This equation for a complex differential form Φ on a pseudo-Riemannian manifold M and for a free field system is given by (Kähler, 1962)

$$
\not{d}\Phi = m\Phi \tag{5}
$$

where $\not{d}$ is the Kähler operator (sometimes called the Hodge–de-Rham operator) and is defined by

$$
\not{d} = d - \delta = d - \star d \star \tag{6}
$$

In this paper the theory to be discussed is formulated in terms of differential forms on space-time M and intrinsic operations constructed from the maps d: $\Lambda^p(M) \rightarrow \Lambda^{p+1}(M)$ and $\star$: $\Lambda^p(M) \rightarrow \Lambda^{4-p}(M)$, where d denotes the exterior derivative, $\Lambda^p(M)$ is the space of p-forms on M, and $\star$ the Hodge map defined with respect to the pseudo-Riemannian metric g of T_xM, the tangent space at the point x of M. Unlike d and δ separately, $\not{d}$ is not a homogeneous operator on differential forms; whereas d increases the degree of a form by one, δ decreases the degree by one. However, $\not{d}^2$ is homogeneous

$$d\!\!\!/^2 = (d - \delta)(d - \delta) = -(d\delta + \delta d) \tag{7}$$

where $d^2 = \delta^2 = 0$. It is readily checked that the Kähler equation iterates to the Klein–Gordon equation, i.e.,

$$\Box\Phi = m^2\Phi \tag{8}$$

where $\Box \equiv -(d\delta + \delta d)$ is the Laplace–Beltrami operator, which preserves the degree of a form $\Box$: $\Lambda^p(M) \to \Lambda^p(M)$. The next task is to make use of the isomorphism between Clifford and the exterior algebras, which will enable us to write the Kähler equation in a form that could perturbatively decouple and expand its solution as a power series in the geodesic distance.

Lemma 1. For a torsion-free connection,

$$d = e^\mu \wedge \nabla_{X_\mu}$$

Proof:

$$\begin{aligned} e^\mu \wedge \nabla_{X_\mu} e^\nu &= e^\mu \wedge \Omega_\xi{}^\nu(X_\mu) e^\xi \\ &= e^\mu \wedge i_{\widetilde{e^\mu}} \wedge \Omega_\xi{}^\nu e^\xi \\ &= \Omega_\xi{}^\nu e^\xi \\ &= de^\nu \end{aligned}$$

where the Levi-Civita orthonormal connection 1-form $\Omega_\nu{}^\mu$ is defined by $\nabla_{X_\mu} e^\nu = \Omega_\xi{}^\nu(X_\mu)e^\xi$.

Lemma 2:

$$\delta = -i_{\widetilde{e^\mu}} \nabla_{X_\mu}$$

Proof:

$$\begin{aligned} \delta &\equiv \star d \star \\ &= \star\left(e^\mu \wedge \nabla_{X_\mu} \star\right) \qquad \text{(from Lemma 1)} \\ &= \star\left(\eta \nabla_{X_\mu} \star \wedge e^\mu\right) \\ &= i_{\widetilde{e^\mu}} \star \eta \nabla_{X_\mu} \star \\ &= -i_{\widetilde{e^\mu}} \nabla_{X_\mu} \qquad (\text{since } \star\star = -\eta \text{ and } \eta^2 = 1) \end{aligned}$$

where the involution η is defined by $\eta X = -X$ if X is a covector. Some of the properties (Talebaoui, 1994) of the involution η are

$$\eta(\alpha \wedge \beta) = \eta\alpha \wedge \eta\beta, \qquad \eta(\alpha \vee \beta) = \eta\alpha \vee \eta\beta, \qquad \forall\alpha, \beta \in \Lambda(M) \tag{9}$$

Here $\Lambda(M) \equiv \cup_{p=0}^{4} \Lambda^p(M)$ is the inhomogeneous differential form. In addition to the exterior algebra over $\Lambda(M)$, one can define the associative Clifford algebra $C[\Lambda^1(M), g]$ associative with g. If $\alpha, \beta \in \Lambda^1(M)$, then their Clifford product denoted by $\alpha \vee \beta$ satisfies

$$\alpha \vee \beta + \beta \vee \alpha = 2g(\tilde{\alpha}, \tilde{\beta}) \tag{10}$$

and the isomorphism between $C[\Lambda^1(M), g]$ and the exterior algebra over $\Lambda(M)$ is related by

$$\alpha\vee = \alpha \wedge + i_{\tilde{\alpha}} \tag{11}$$

Hence if $\alpha, \beta \in \Lambda^1(M)$, then

$$\alpha \vee \beta = \alpha \wedge \beta + i_{\tilde{\alpha}}\beta = \alpha \wedge \beta + g(\tilde{\alpha}, \tilde{\beta}) \tag{12}$$

Lemma 3:

$$\not{d} = e^{\mu} \vee \nabla_{X_\mu}$$

Proof:

$$\begin{aligned} e^{\mu} \vee \nabla_{X_\mu} &= e^{\mu} \wedge \nabla_{X_\mu} + i_{\widetilde{e^{\mu}}}\nabla_{X_\mu} \qquad \text{[from (12)]} \\ &= d - \delta \qquad \text{(from Lemmas 1 and 2)} \\ &= \not{d} \end{aligned}$$

Using Lemma 3, we can write the Kähler equation (5) as

$$e^{\mu} \vee \nabla_{X_\mu}\Phi = m\Phi \tag{13}$$

Now we write

$$\Phi = \sum_{i,j} \Phi_{ij} f_{ij} \tag{14}$$

and define

$$\nabla_{X_\mu} f_{ij} \equiv \psi_{ij}^{i'j'}(X_\mu) f_{i'j'} \tag{15}$$

$$e^{\mu} \vee f_{ij} \equiv \gamma_{ii'}^{\mu} f_{i'j} \tag{16}$$

where Φ_{ij} are real-valued functions on M, and f_{ij} is the matrix basis.

$\psi_{ij}^{i'j'}(X_\mu)$ play the role of the connection coefficients, and $\gamma_{ii'}^{\mu}$ is a constant. Using the Kähler equation given in the form (13), we get

$$e^{\mu} \vee \left(\sum_{ij} (X_\mu \Phi_{ij}) f_{ij} + \sum_{ij} \Phi_{ij} \nabla_{X_\mu} f_{ij} \right) = m \sum_{ij} \Phi_{ij} f_{ij} \tag{17}$$

or

$$\left.\begin{array}{ll} (X_\mu \alpha_i)\gamma_{ir}^{\mu} + V_r^1 = m\alpha_r, & (X_\mu \beta_i)\gamma_{ir}^{\mu} + V_r^2 = m\beta_r \\ (X_\mu \chi_i)\gamma_{ir}^{\mu} + V_r^3 = m\chi_r, & (X_\mu \rho_i)\gamma_{ir}^{\mu} + V_r^4 = m\rho_r \end{array}\right\} \tag{18}$$

where

$$\alpha_i \equiv \Phi_{i1}, \qquad \beta_i \equiv \Phi_{i2}, \qquad \chi_i \equiv \Phi_{i3}, \qquad \rho_i \equiv \Phi_{i4} \tag{19}$$

and

$$V_r^p \equiv [\alpha_i \psi_{i1}^{i'p}(X_\mu) + \beta_i \psi_{i2}^{i'p}(X_\mu) + \chi_i \psi_{i3}^{i'p}(X_\mu) + \rho_i \psi_{i4}^{i'p}(X_\mu)]\gamma_{i'r}^{\mu} \tag{20}$$

The term V_r^p represents the interaction term, and in the case when it is switched off, the Kähler equation (18) for the four spinors decouples into four copies of the Dirac equation. It is worth mentioning here that the Kähler equation might be more appropriate when discussing spinor fields and their relation to gravitation. An arbitrary Kähler field Φ on space-time has 16 complex components, whereas a Dirac spinor of the complexified Clifford algebra has only four complex components. Thus a general solution of the Kähler equation has more degrees of freedom than a solution to the Dirac equation. This raises the question of the significance of using the Kähler equation for the description of particles in nature such as the electron–positron field that are conventionally described by the Dirac equation.

As the real Clifford algebra, $C(M)$ is isomorphic to a real 4×4 matrix algebra. It may be decomposed into four minimal left ideals characterized by a complete set of four minimal rank (*primitive*) idempotent projectors (P_j) such that

$$P_i \vee P_j = P_i \delta_{ij} \qquad i \text{ not summed} \tag{21}$$

i.e., they are pairwise (*orthogonal*) under Clifford multiplication $\vee$. The idempotent projectors satisfying (21) are given by

$$\begin{array}{ll} P_1 = \dfrac{1}{4}[(1 + e^{02})(1 + e^1)], & P_2 = \dfrac{1}{4}[(1 - e^{02})(1 + e^1)] \\ P_3 = \dfrac{1}{4}[(1 + e^{02})(1 - e^1)], & P_4 = \dfrac{1}{4}[(1 - e^{02})(1 - e^1)] \end{array} \tag{22}$$

One can easily check that the above projectors satisfy (21), and also $\Sigma_{i=1}^{4} P_i$

Table I. The Matrix Basis f_{ij} Constructed from the Four Projectors

	f_{ij}			
	$j = 1$	$j = 2$	$j = 3$	$j = 4$
$i = 1$	P_1	$P_1 e^{03}$	$P_1 e^3$	$P_1 e^0$
$i = 2$	$P_2 e^{03}$	P_2	$P_2 e^0$	$P_2 e^3$
$i = 3$	$P_3 e^3$	$P_3 e^0$	P_3	$P_3 e^{03}$
$i = 4$	$P_4 e^0$	$P_4 e^3$	$P_4 e^{03}$	P_4

$= 1$. If we denote by $S_p(\Phi)$ the p-form component of Φ, then the primitivity of each P_j implies

$$P_j \vee \Phi \vee P_j = 4S_0(\Phi \vee P_j)P_j \tag{23}$$

where the S_0 projects the 0-form out of the parenthesis. Since $C(M)$ is a total matrix algebra, it is possible to construct a basis $f_{ij} \in \Gamma[C(M)]$, $i, j = 1, 2, 3, 4$, that satisfies

$$f_{ij} \vee f_{ji'} = f_{ii'} \quad \text{no } j \text{ sum implied} \qquad \text{and} \qquad f_{ij} \vee f_{i'j'} = 0 \qquad \forall j \neq i' \tag{24}$$

that is, the f_{ij} have the algebra of an ordinary matrix basis. Now the left-hand side of (15) can be written as

$$\nabla_{X_\mu} f_{ij} = \sum_{i',j'} 4S_0(f_{j'i'} \nabla_{X_\mu} f_{ij}) f_{i'j'} \tag{25}$$

The basis f_{ij} constructed from the four idempotent projectors satisfying (24) are given in Table I.

Since $e^\mu \vee f_{ij} \equiv \gamma^\mu_{ii'} f_{i'j}$,

$$e^1 \vee f_{11} = \gamma^1_{1i'} f_{i'1} = \gamma^1_{11} f_{11} + \gamma^1_{12} f_{21} + \gamma^1_{13} f_{31} + \gamma^1_{14} f_{41} \tag{26}$$

or $\gamma^1_{11} = 1$ and $\gamma^1_{12} = \gamma^1_{13} = \gamma^1_{14} = 0$ because $e^1 \vee f_{11} = f_{11}$. Similarly, when $i = 2$, then $\gamma^1_{22} = 1$ and $\gamma^1_{21} = \gamma^1_{23} = \gamma^1_{24} = 0$. For $i = 3$, we have $\gamma^1_{33} = -1$ and $\gamma^1_{31} = \gamma^1_{32} = \gamma^1_{34} = 0$. Finally, when $i = 4$, we get $\gamma^1_{44} = -1$ and $\gamma^1_{41} = \gamma^1_{42} = \gamma^1_{43} = 0$. Therefore, when $\mu = 1$, then $\gamma^1_{ii'}$ can be written as a 4×4 matrix, i.e.,

$$\gamma^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{27}$$

By following the same procedure in obtaining γ^1, we get

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\gamma^0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{28}$$

The above matrices satisfy

$$\left.\begin{aligned} \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu &= 2\eta^{\mu\nu} \\ \gamma^{0t} = -\gamma^0 \qquad & (\gamma^n)^2 = 1 \\ \gamma^{nt} = \gamma^n \qquad & (\gamma^0)^2 = -1 \end{aligned}\right\} \tag{29}$$

where $n = 1, 2, 3$ and t denotes transpose of the matrix.

4. THE β SPINORS IN RNC

The aim of this section is to obtain the dynamic equations for the Kähler spinors β_i in RNC when they are subjected to the interacting term V_r^2. The connection 1-form (to order of t^2) in terms of RNC is given by

$$\Omega^\mu{}_\nu = \frac{t^2}{2} \hat{R}^\mu{}_{\nu kl}\xi^k d\xi^l \tag{30}$$

where $\hat{R}^\mu{}_{\nu kl} \equiv R^\mu{}_{\nu kl}(t = 0) \in \Lambda^0(M)$ and $\Omega^\mu{}_\nu \in \Lambda^1(M)$. Making use of (4), we can compute the connection coefficients in RNC; for example, let us find $\Omega^1{}_2(X_1)$. From (30), we obtain

$$\Omega^1{}_2(X_1) = \frac{t^2}{2} \hat{R}^1{}_{2kl}\xi^k d\xi^l(X_1) \tag{31}$$

or

$$\Omega^1{}_2(X_1) = \frac{t^2}{2} [\hat{R}^1{}_{2k0}\xi^k d\xi^0(X_1) + \hat{R}^1{}_{2k\underline{l}}\xi^k d\xi^{\underline{l}}(X_1)] \tag{32}$$

Table II. $(24/Rt)\Omega_{\mu\nu}(X_1)$

	$(24/Rt)\Omega_{\mu\nu}(X_1)$			
	$\nu = 0$	$\nu = 1$	$\nu = 2$	$\nu = 3$
$\mu = 0$	0	ξ^0	0	0
$\mu = 1$	$-\xi^0$	0	$-\xi^2$	$-\xi^3$
$\mu = 2$	0	ξ^2	0	0
$\mu = 3$	0	ξ^3	0	0

where $l = 0$, $\underline{l}$ and $\underline{l} = 1, 2, 3$. Using (4) implies that

$$\begin{aligned}\Omega_{12}(X_1) &= \frac{t^2}{2}\left[\hat{R}^1{}_{2k0}\xi^k\left(\frac{\lambda\xi^0}{t}\right)\xi^1 + \hat{R}^1{}_{2k\underline{l}}\xi^k\frac{1}{t}(\delta^{1\underline{l}} + \lambda\xi^{\underline{l}}\xi^1)\right] \\ &= \frac{t}{2}[\hat{R}^1{}_{2k0}\xi^k\lambda\xi^0\xi^1 + \hat{R}^1{}_{2kl}\xi^k + \hat{R}^1{}_{2k\underline{l}}\xi^k\lambda\xi^{\underline{l}}\xi^1]\end{aligned} \tag{33}$$

In this problem we restrict ourselves to the space-time of constant curvature because it is the only one for which we managed to find spinorial solution to the Kähler spinors β_i. The space-time metrics of constant curvature are characterized by the condition (Hawking and Ellis, 1973)

$$R^{\mu\nu}{}_{kl} = \frac{R}{12}(\delta^{\mu}_{k}\delta^{\nu}_{l} - \delta^{\mu}_{l}\delta^{\nu}_{k}) \tag{34}$$

Using (34), we find that equation (33) becomes

$$\begin{aligned}\Omega_{12}(X_1) &= \frac{Rt}{24}[(\delta^1_k\delta^2_0 - \delta^1_0\delta^2_k)\xi^k\lambda\xi^0\xi^1 + (\delta^1_k\delta^2_1 - \delta^1_1\delta^2_k)\xi^k + (\delta^1_k\delta^2_{\underline{l}} - \delta^1_{\underline{l}}\delta^2_k)\xi^k\lambda\xi^{\underline{l}}\xi^1] \\ &= -\frac{Rt}{24}\xi^2\end{aligned} \tag{35}$$

Similarly, one can compute all the remaining 63 connection coefficients in RNC. The results of these computations are given in Tables II–V.

Table III. $(24/Rt)\Omega_{\mu\nu}(X_2)$

	$(24/Rt)\Omega_{\mu\nu}(X_2)$			
	$\nu = 0$	$\nu = 1$	$\nu = 2$	$\nu = 3$
$\mu = 0$	0	0	ξ^0	0
$\mu = 1$	0	0	ξ^1	0
$\mu = 2$	$-\xi^0$	$-\xi^1$	0	$-\xi^3$
$\mu = 3$	0	0	ξ^3	0

Table IV. $(24/Rt)\Omega_{\mu\nu}(X_3)$

	$(24/Rt)\Omega_{\mu\nu}(X_3)$			
	$\nu = 0$	$\nu = 1$	$\nu = 2$	$\nu = 3$
$\mu = 0$	0	0	0	ξ^0
$\mu = 1$	0	0	0	ξ^1
$\mu = 2$	0	0	0	ξ^2
$\mu = 3$	$-\xi^0$	$-\xi^1$	$-\xi^2$	0

Table V. $(24/Rt)\Omega_{\mu\nu}(X_0)$

	$(24/Rt)\Omega_{\mu\nu}(X_0)$			
	$\nu = 0$	$\nu = 1$	$\nu = 2$	$\nu = 3$
$\mu = 0$	0	$-\xi^1$	$-\xi^2$	$-\xi^3$
$\mu = 1$	ξ^1	0	0	0
$\mu = 2$	ξ^2	0	0	0
$\mu = 3$	ξ^3	0	0	0

From Tables II–V it is clear that the connection coefficients are functions of the geodesic distance and the geodesic coordinates, and if we define

$$\psi_{ij}^{i'j'}(X_\mu) \equiv t\overline{\psi}_{ij}^{i'j'}(X_\mu) \tag{36}$$

In this paper we shall seek a spinorial solution to the β_i spinors. From (18) we have

$$(X_\mu\beta_i)\gamma_{ir}^{\mu} + t\overline{V}_r^2 = m\beta_r \tag{37}$$

where

$$\overline{V}_r^2 \equiv [\alpha_i\overline{\psi}_{i1}^{i'2}(X_\mu) + \beta_i\overline{\psi}_{i2}^{i'2}(X_\mu) + \chi_i\overline{\psi}_{i3}^{i'2}(X_\mu) + \rho_i\overline{\psi}_{i4}^{i'2}(X_\mu)]\gamma_{i'r}^{\mu} \tag{38}$$

When $\overline{V}_r^2 \neq 0$, we shall expand the solution in a power series of the geodesic distance t around $t = 0$, so we assume that, for some Minkowski four-vector k,

$$\beta_i(\xi, t) = \beta_i^{(0)}(k) + t\beta_i^{(1)}(\xi, k) + t^2\beta_i^{(2)}(\xi, k) + t^3\beta_i^{(3)}(\xi, k) + \cdots \tag{39}$$

with $k^\mu k_\mu = -m^2$. For simplicity we shall define $\beta_i^{(0)} \equiv \beta_i^{(0)}(k)$, $\beta_i^{(1)} \equiv \beta_i^{(1)}(\xi, k)$, and $\beta_i^{(2)} \equiv \beta_i^{(2)}(\xi, k)$. Here $\beta_i^{(0)}(k)$ are arbitrary constant spinors which satisfy the flat space-time Dirac equation for a particle with mass m.

Using (3), we find that equation (37) gives

$$
\begin{aligned}
&-\lambda\xi^1[\beta_i^{(1)} + 2t\beta_i^{(2)} + 3t^2\beta_i^{(3)}]\gamma_{ir}^1 + \left[\frac{\partial\beta_i^{(1)}}{\partial\xi^1} + t\frac{\partial\beta_i^{(2)}}{\partial\xi^1} + t^2\frac{\partial\beta_i^{(3)}}{\partial\xi^1}\right]\gamma_{ir}^1 \\
&\quad + \lambda\xi^0\xi^l\left[\frac{\partial\beta_i^{(1)}}{\partial\xi^l} + t\frac{\partial\beta_i^{(2)}}{\partial\xi^l} + t^2\frac{\partial\beta_i^{(3)}}{\partial\xi^l}\right]\gamma_{ir}^1 - \lambda\xi^2[\beta_i^{(1)} + 2t\beta_i^{(2)} + 3t^2\beta_i^{(3)}]\gamma_{ir}^2 \\
&\quad + \left[\frac{\partial\beta_i^{(1)}}{\partial\xi^2} + t\frac{\partial\beta_i^{(2)}}{\partial\xi^2} + t^2\frac{\partial\beta_i^{(3)}}{\partial\xi^2}\right]\gamma_{ir}^2 + \lambda\xi^2\xi^l\left[\frac{\partial\beta_i^{(1)}}{\partial\xi^l} + t\frac{\partial\beta_i^{(2)}}{\partial\xi^l} + t^2\frac{\partial\beta_i^{(3)}}{\partial\xi^l}\right]\gamma_{ir}^2 \\
&\quad - \lambda\xi^3[\beta_i^{(1)} + 2t\beta_i^{(2)} + 3t^2\beta_i^{(3)}]\gamma_{ir}^3 + \left[\frac{\partial\beta_i^{(1)}}{\partial\xi^3} + t\frac{\partial\beta_i^{(2)}}{\partial\xi^2} + t^2\frac{\partial\beta_i^{(3)}}{\partial\xi^3}\right]\gamma_{ir}^3 \\
&\quad + \lambda\xi^3\xi^l\left[\frac{\partial\beta_i^{(1)}}{\partial\xi^l} + t\frac{\partial\beta_i^{(2)}}{\partial\xi^l} + t^2\frac{\partial\beta_i^{(3)}}{\partial\xi^l}\right]\gamma_{ir}^3 + \lambda\xi^0[\beta_i^{(1)} + 2t\beta_i^{(2)} + 3t^2\beta_i^{(3)}]\gamma_{ir}^0 \\
&\quad - \lambda\xi^0\xi^l\left[\frac{\partial\beta_i^{(1)}}{\partial\xi^l} + t\frac{\partial\beta_i^{(2)}}{\partial\xi^l} + t^2\frac{\partial\beta_i^{(3)}}{\partial\xi^l}\right]\gamma_{ir}^0 \\
&\quad + \{(\alpha_i^{(0)} + t\alpha_i^{(1)} + t^2\alpha_i^{(2)} + \cdots)t\bar{\psi}_{i1}^{\prime 2}(X_\mu) \\
&\quad + (\beta_i^{(0)} + t\beta_i^{(1)} + t^2\beta_i^{(2)} + \cdots)t\bar{\psi}_{i2}^{\prime 2}(X_\mu) \\
&\quad + (\chi_i^{(0)} + t\chi_i^{(1)} + t^2\chi_i^{(2)} + \cdots)t\bar{\psi}_{i3}^{\prime 2}(X_\mu) \\
&\quad + (\rho_i^{(0)} + t\rho_i^{(1)} + t^2\rho_i^{(2)} + \cdots)t\bar{\psi}_{i4}^{\prime 2}(X_\mu)\}\gamma_{i'r}^{\mu} \\
&= m[\beta_r^{(0)} + t\beta_r^{(1)} + t^2\beta_r^{(2)} + t^3\beta_r^{(3)}]
\end{aligned} \tag{40}
$$

Equation (40) is the Kähler equation for the β_i spinors in RNC. This work is to order t^2; therefore we shall not consider any higher iterations of $\beta_i^{(3)}$. Then by equating like powers of t in (40), we obtain

$$\mathcal{D}\beta_i^{(1)} = m\beta_r^{(0)} \tag{41}$$

where

$$
\begin{aligned}
\mathcal{D} \equiv &\left[\frac{\partial}{\partial\xi^1} + \lambda\xi^1\xi^l\frac{\partial}{\partial\xi^l} - \lambda\xi^1\right]\gamma^1 + \left[\frac{\partial}{\partial\xi^2} + \lambda\xi^2\xi^l\frac{\partial}{\partial\xi^l} - \lambda\xi^2\right]\gamma^2 \\
&+ \left[\frac{\partial}{\partial\xi^3} + \lambda\xi^3\xi^l\frac{\partial}{\partial\xi^l} - \lambda\xi^3\right]\gamma^3 + \left[\lambda\xi^0 - \lambda\xi^0\xi^l\frac{\partial}{\partial\xi^l}\right]\gamma^0
\end{aligned} \tag{42}
$$

and similarly

$$\mathcal{D}\beta_i^{(2)} = \mathcal{Q}_r \tag{43}$$

where

$$\mathcal{Q}_r \equiv m\beta_r^{(1)} - \overline{V}_r^2(0) \tag{44}$$

and

$$\begin{aligned}\overline{V}_r^2(0) \equiv [\alpha_i^{(0)}\overline{\psi}_{i1}^{i'2}(X_\mu) + \beta_i^{(0)}\overline{\psi}_{i2}^{i'2}(X_\mu) + \chi_i^{(0)}\overline{\psi}_{i3}^{i'2}(X_\mu) \\ + \rho_i^{(0)}\overline{\psi}_{i4}^{i'2}(X_\mu)]\gamma_{i'r}^{\mu}\end{aligned} \tag{45}$$

5. SPINORIAL SOLUTION TO THE KÄHLER EQUATION

Now let us examine the solutions to the β_i Kähler spinor in space-time of constant curvature. Define

$$\tilde{F}^n \equiv \left[\frac{\partial}{\partial \xi^n} + \lambda \xi^n \xi^l \frac{\partial}{\partial \xi^l} - \lambda \xi^n\right] \tag{46}$$

Since $\gamma_{ir}^n = \gamma_{ri}^n$ and $\gamma_{ir}^0 = -\gamma_{ri}^0$, equation (41) can be written as

$$\left[\gamma_{ri}^1\tilde{F}^1 + \gamma_{ri}^2\tilde{F}^2 + \gamma_{ri}^3\tilde{F}^3 - \gamma_{ri}^0\left(\frac{\partial}{\partial\xi^0} - \tilde{F}^0\right)\right]\beta_i^{(1)} = m\beta_r^{(0)} \tag{47}$$

Now by making use of the γ matrices (27) and (28), we find that equation (47) becomes

$$\left.\begin{aligned}
&\tilde{F}^1\beta_1^{(1)} + \tilde{F}^3\beta_3^{(1)} - \tilde{F}^2\beta_4^{(1)} + \left(\lambda\xi^0\xi^l\frac{\partial}{\partial\xi^l} - \lambda\xi^0\right)\beta_4^{(1)} = m\beta_1^{(0)} \\
&\tilde{F}^1\beta_2^{(1)} - \tilde{F}^2\beta_3^{(1)} - \left(\lambda\xi^0\xi^l\frac{\partial}{\partial\xi^l} - \lambda\xi^0\right)\beta_3^{(1)} - \tilde{F}^3\beta_4^{(1)} = m\beta_2^{(0)} \\
&\tilde{F}^3\beta_1^{(1)} + \left(\lambda\xi^0\xi^l\frac{\partial}{\partial\xi^l} - \lambda\xi^0\right)\beta_2^{(1)} - \tilde{F}^2\beta_2^{(1)} - \tilde{F}^1\beta_3^{(1)} = m\beta_3^{(0)} \\
&\left(\lambda\xi^0\xi^l\frac{\partial}{\partial\xi^l} - \lambda\xi^0\right)\beta_1^{(1)} + \tilde{F}^2\beta_1^{(1)} + \tilde{F}^3\beta_2^{(1)} + \tilde{F}^1\beta_4^{(1)} = -m\beta_4^{(0)}
\end{aligned}\right\} \tag{48}$$

Equation (48) represents four partial differential equations for the first iterations of the spinors $\beta_1^{(1)}, \ldots, \beta_4^{(1)}$, and each of these spinors is a function of the geodesic coordinates. By a process of elimination among these four equations, we can solve for $\beta_1^{(1)}$, $\beta_2^{(1)}$, $\beta_3^{(1)}$, and $\beta_4^{(1)}$ individually as follows:

$$\left.\begin{aligned}
\beta_1^{(1)} &= \frac{m}{4}\,[\xi^1\beta_1^{(0)} + \xi^3\beta_3^{(0)} + (\xi^0 - \xi^2)\beta_4^{(0)}] \\
\beta_2^{(1)} &= -\frac{m}{4}\,[-\xi^1\beta_2^{(0)} + \xi^3\beta_4^{(0)} + (\xi^2 + \xi^0)\beta_3^{(0)}] \\
\beta_3^{(1)} &= \frac{m}{4}\,[\xi^3\beta_1^{(0)} - \xi^1\beta_3^{(0)} + (\xi^0 - \xi^2)\beta_2^{(0)}] \\
\beta_4^{(1)} &= -\frac{m}{4}\,[\xi^1\beta_4^{(0)} + \xi^3\beta_2^{(0)} + (\xi^0 + \xi^2)\beta_1^{(0)}]
\end{aligned}\right\} \tag{49}$$

Equation (49) represents the first-order spinors $\beta_i^{(1)}$ and we notice that the solutions are written in terms of the $\beta_i^{(0)}$ spinor components and excluding any mixture of the remaining spinors. In the massless case these spinors vanish and the Kähler field propagates as though in Minkowski space-time.

Now we define

$$F^n \equiv \left[\frac{\partial}{\partial\xi^n} + \lambda\xi^n\xi^{\underline{l}}\frac{\partial}{\partial\xi^{\underline{l}}} - 2\lambda\xi^n\right] \tag{50}$$

Here $n = 0, 1, 2, 3$ and $\underline{l} = 1, 2, 3$. If we assume that $R = 12m^2$ and use equations (44) and (45), the $\beta_i^{(2)}$ spinors (43) give rise to the following system of four coupled partial differential equations:

$$\begin{aligned}
&F^1\beta_1^{(2)} + F^3\beta_3^{(2)} - F^2\beta_4^{(2)} + \left\{\lambda\xi^0\xi^{\underline{l}}\frac{\partial}{\partial\xi^{\underline{l}}} - 2\lambda\xi^0\right\}\beta_4^{(2)} \\
&\quad = M[(\xi^2 + \xi^0)(\alpha_3^{(0)} - \chi_1^{(0)}) + (\xi^1 + \xi^3)\alpha_4^{(0)} + 2\xi^1(\beta_1^{(0)} - \chi_4^{(0)}) \\
&\qquad + 2\xi^3\beta_3^{(0)} - (\xi^0 + 3\xi^2)\beta_4^{(0)} - \xi^3\rho_1^{(0)} + \xi^1\rho_3^{(0)}]
\end{aligned} \tag{51}$$

$$\begin{aligned}
&F^1\beta_2^{(2)} - F^2\beta_3^{(2)} - \left\{\lambda\xi^0\xi^{\underline{l}}\frac{\partial}{\partial\xi^{\underline{l}}} - 2\lambda\xi^0\right\}\beta_3^{(2)} - F^3\beta_4^{(2)} \\
&\quad = M[(\xi^3 - \xi^1)\alpha_3^{(0)} - (\xi^2 + \xi^0)(\alpha_4^{(0)} + \chi_2^{(0)}) + 2\xi^1\beta_1^{(0)} \\
&\qquad + (3\xi^0 - \xi^2)\beta_3^{(0)} - 2\xi^3\beta_4^{(0)} - \xi^3\rho_2^{(0)} - \xi^1\rho_4^{(0)}]
\end{aligned} \tag{52}$$

$$\begin{aligned}
&F^3\beta_1^{(2)} + \left\{\lambda\xi^0\xi^{\underline{l}}\frac{\partial}{\partial\xi^{\underline{l}}} - 2\lambda\xi^0\right\}\beta_2^{(2)} - F^2\beta_2^{(2)} - F^1\beta_3^{(2)} \\
&\quad = M[(\xi^2 + \xi^0)(\alpha_1^{(0)} + \chi_3^{(0)}) + (\xi^1 + \xi^3)\alpha_2^{(0)} + 2\xi^3\beta_1^{(0)} \\
&\qquad + (\xi^0 - 3\xi^2)\beta_2^{(0)} - 2\xi^1(\beta_3^{(0)} + \chi_2^{(0)}) + \xi^1\rho_1^{(0)} + \xi^3\rho_3^{(0)}]
\end{aligned} \tag{53}$$

$$\left\{\lambda\xi^0\xi^l\frac{\partial}{\partial\xi^l} - 2\lambda\xi^0\right\}\beta_1^{(2)} + F^2\beta_1^{(2)} + F^3\beta_2^{(2)} + F^1\beta_4^{(2)}$$

$$= M[(\xi^1 - \xi^3)\alpha_1^{(0)} + (\xi^2 + \xi^0)(\alpha_2^{(0)} - \chi_4^{(0)})$$

$$+ (\xi^2 - 3\xi^0)\beta_1^{(0)} + 2\xi^3\beta_2^{(0)} + 2\xi^1\beta_4^{(0)} + \xi^1\rho_2^{(0)} - \xi^3\rho_4^{(0)}] \tag{54}$$

where $M = R/48$ and solutions to (51)–(54) are given by

$$\beta_1^{(2)} = B\Bigg[\left\{2\xi^1\xi^2 + \frac{1}{2}\xi^2\xi^3 + \frac{9}{2}\xi^3\xi^0 - 2\xi^1\xi^0\right\}\alpha_1^{(0)}$$

$$+ \left\{(\xi^2)^2 + 2\xi^1\xi^3 + (\xi^3)^2 - (\xi^0)^2 - \frac{3}{\lambda}\right\}\alpha_2^{(0)} + \frac{5}{2}\xi^1(\xi^2 + \xi^0)\alpha_3^{(0)}$$

$$+ \left\{(\xi^1)^2 + 2\xi^1\xi^3 - \frac{1}{\lambda}\right\}\alpha_4^{(0)} + \left\{3(\xi^1)^2 + \frac{4}{3}(\xi^2)^2 + 3(\xi^3)^2\right.$$

$$\left.+ \frac{16}{3}(\xi^0)^2 - \frac{20}{3}\xi^2\xi^0 - \frac{2}{\lambda}\right\}\beta_1^{(0)} - \frac{5}{3}\xi^3(\xi^2 + \xi^0)\beta_2^{(0)}$$

$$- \xi^1\left(\frac{5}{3}\xi^2 + 5\xi^0\right)\beta_4^{(0)}$$

$$- 10\xi^1(\xi^2 + \xi^0)\chi_1^{(0)} - \frac{32}{7}\xi^1\xi^3\chi_2^{(0)} + \left\{\frac{4}{\lambda} - \frac{16}{7}(\xi^1)^2\right.$$

$$\left.- \frac{6}{7}(\xi^2)^2 + \frac{6}{7}(\xi^0)^2\right\}\chi_4^{(0)}$$

$$+ 2\xi^1(\xi^2 - \xi^0)\rho_2^{(0)} + \left\{(\xi^1)^2 + (\xi^3)^2 - \frac{2}{\lambda}\right\}\rho_3^{(0)} + 2\xi^3(\xi^0 - \xi^2)\rho_4^{(0)}\Bigg] \tag{55}$$

$$\beta_2^{(2)} = B\Bigg[\left\{\frac{1}{\lambda} - \frac{5}{4}(\xi^2)^2 - (\xi^3)^2 - \frac{5}{4}(\xi^0)^2 + 2\xi^1\xi^3 - \frac{5}{2}\xi^2\xi^0\right\}\alpha_1^{(0)}$$

$$+ \left\{2\xi^1\xi^3 - (\xi^1)^2 + \frac{1}{\lambda}\right\}\alpha_3^{(0)} - 2\xi^1(\xi^2 + \xi^0)\alpha_4^{(0)}$$

$$+ \frac{5}{3}(\xi^2 + 5\xi^0)(\xi^1\beta_3^{(0)} - \xi^3\beta_1^{(0)})$$

$$+ \left\{2(\xi^1)^2 + \frac{11}{3}(\xi^2)^2 + 2(\xi^3)^2 - \frac{1}{3}(\xi^0)^2 + \frac{10}{3}\xi^2\xi^0 - \frac{8}{\lambda}\right\}\beta_2^{(0)}$$

$$+ \frac{10}{7}\xi^1(\xi^2+\xi^0)\chi_2^{(0)} - \frac{12}{7}\xi^3(\xi^2+\xi^0)\chi_4^{(0)} - 2\xi^1(\xi^2+\xi^0)(\rho_1^{(0)}+\alpha_2^{(0)})$$

$$- 2\xi^3(\xi^2+\xi^0)\rho_3^{(0)} + \left\{\frac{2}{\lambda} - (\xi^1)^2 - (\xi^3)^2\right\}\rho_4^{(0)}\Bigg] \tag{56}$$

$$\beta_3^{(2)} = B\Bigg[-\frac{5}{2}\xi^1(\xi^2+\xi^0)\alpha_1^{(0)} + \left\{\frac{1}{\lambda} - (\xi^1)^2 - 2\xi^1\xi^3\right\}\alpha_2^{(0)}$$

$$+ \left\{\frac{1}{2}\xi^2\xi^3 + \frac{9}{2}\xi^3\xi^0 - 2\xi^1\xi^0 + 2\xi^1\xi^2\right\}\alpha_3^{(0)}$$

$$+ \left\{(\xi^2)^2 + (\xi^3)^2 - (\xi^0)^2 + 2\xi^1\xi^3 - \frac{3}{\lambda}\right\}\alpha_4^{(0)} + \frac{5}{3}\xi^1(\xi^2+\xi^0)\beta_2^{(0)}$$

$$+ \left\{3(\xi^1)^2 + \frac{4}{3}(\xi^2)^2 + 3(\xi^3)^2 + \frac{16}{3}(\xi^0)^2 - \frac{20}{3}\xi^2\xi^0 - \frac{2}{\lambda}\right\}\beta_3^{(0)}$$

$$- \xi^3\left(\frac{5}{3}\xi^2 + 5\xi^0\right)\beta_4^{(0)} + \left\{\frac{16}{7}(\xi^1)^2 + \frac{6}{7}(\xi^2)^2 - \frac{6}{7}(\xi^0)^2 - \frac{4}{\lambda}\right\}\chi_2^{(0)}$$

$$- 10\xi^1(\xi^2+\xi^0)\chi_3^{(0)}$$

$$- \frac{32}{7}\xi^1\xi^3\chi_4^{(0)} + \left\{\frac{2}{\lambda} - (\xi^1)^2 - (\xi^3)^2\right\}\rho_1^{(0)} + 2\xi^3(\xi^2-\xi^0)\rho_2^{(0)}$$

$$+ 2\xi^1(\xi^2-\xi^0)\rho_4^{(0)}\Bigg] \tag{57}$$

$$\beta_4^{(2)} = B\Bigg[2\xi^1(\xi^2+\xi^0)\alpha_2^{(0)} + \Bigg\{2\xi^1\xi^3 - \frac{5}{4}(\xi^2)^2 - (\xi^3)^2 - \frac{5}{2}\xi^2\xi^0$$

$$- \frac{5}{4}(\xi^0)^2 + \frac{1}{\lambda}\Bigg\}\alpha_3^{(0)}$$

$$+ \left\{(\xi^1)^2 - 2\xi^1\xi^3 - \frac{1}{\lambda}\right\}\alpha_1^{(0)} - 2\xi^1(\xi^2+\xi^0)\alpha_4^{(0)}$$

$$- \frac{5}{3}\xi^1(\xi^2+5\xi^0)\beta_1^{(0)} - \frac{5}{3}\xi^3(\xi^2+5\xi^0)\beta_3^{(0)}$$

$$+ \left\{\frac{7}{3}(\xi^1)^2 + 4(\xi^2)^2 + \frac{7}{3}(\xi^3)^2 + \frac{8}{3}(\xi^0)^2 + \frac{20}{3}\xi^2\xi^0 - \frac{6}{\lambda}\right\}\beta_4^{(0)}$$

$$+ \frac{12}{7} \xi^3(\xi^2 + \xi^0)\chi_2^{(0)}$$

$$+ \frac{10}{7} \xi^1(\xi^2 + \xi^0)\chi_4^{(0)} + 2\xi^3(\xi^2 + \xi^0)\rho_1^{(0)} + \left\{(\xi^1)^2 + (\xi^3)^2 - \frac{2}{\lambda}\right\}\rho_2^{(0)}$$

$$- 2\xi^1(\xi^2 + \xi^0)\rho_3^{(0)}\Bigg] \tag{58}$$

where $B = m^2/40$. For the massless case, the $\beta_i^{(1)}$ and $\beta_i^{(2)}$ equations will vanish. Therefore, the only ones surviving are those of zero order, $\beta_i^{(0)}$. The existence of (nonzero) higher iterations $\beta_i^{(1)}$ and $\beta_i^{(2)}$ indicates the presence of a gravitational field, which these equations could be used to study, and the Kähler spinors could be used as a probe of space-time curvature. In the case of the absence of gravitational field, i.e., when the interaction term $V_r^p = 0$, the Kähler spinors propagate as though in Minkowski space-time. Thus the Kähler equation may be decoupled into four sets of equations, one for each four-dimensional minimal left ideal (MLI). Therefore, in the absence of the interaction term in each set [equation (18)] the components of the MLI are coupled in a way that is isomorphic to the coupling between the four components of a spinor that satisfies the Dirac equation. It is worth mentioning here that the Kähler spinors α_i, β_i, χ_i, and ρ_i propagate independently in curved space-time, due to each one of them being subjected to different interaction terms. For instance, the α_i spinor is subjected to a V_r^1, and the β_i spinor will be subjected to a different interaction term, which in this case is V_r^2, and so on for the remaining spinors. This of course will alter the connection coefficients for each individual spinor and in turn gives rise to completely different structure equations describing the spinor in question. Therefore, each spinor will give rise to a different set of partial differential equations which in turn requires different techniques in solving these equations. The only common feature these spinors possess is that in the absence of the interaction term, they will give rise to an identical copy of the Dirac equation.

In conclusion we have shown in this paper how the propagation of the Kähler spinors β_i is affected by the presence of an interaction term in space-time of constant curvature. This has been achieved by making use of the isomorphism between the real Clifford algebra and the total matrix algebra M_4, which enables us to write the Kähler field in terms of a sum of spinors. The framework set up in this paper can serve as a starting point for calculating and extending the work to higher order in the Riemann curvature tensor. This analysis may be of relevance in any model attempting to relate spinor solutions of the Kähler equation to a quantum gravity context.

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